Review: Limit Techniques - 10/14/16

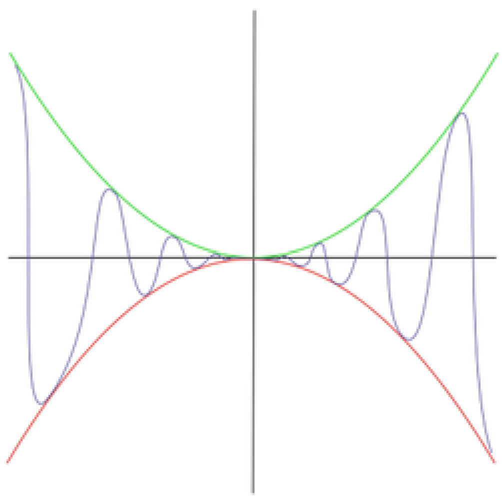
1 Squeeze Theorem

Theorem 1.0.1 *If $f(x) \leq g(x)$ when x is near a (except possibly at a), and the limits of f and g both exist when x approaches a, then*

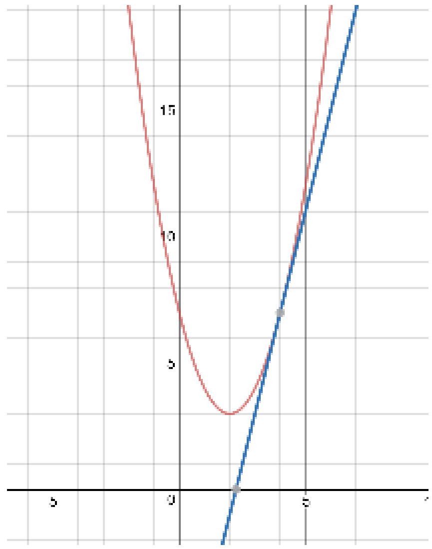
$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Theorem 1.0.2 ***Squeeze Theorem*** *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$.*

Example 1.0.3 *Use the Squeeze Theorem to find $\lim_{x\to 0} x^2 \sin(1/x)$. First we need to find functions that are always smaller and bigger than this one. Let's start with $\sin(1/x)$. We know that $-1 \leq \sin(x) \leq 1$, no matter what x we plug in. So if we plug in $\frac{1}{x}$, this will still hold, so we have $-1 \leq \sin(1/x) \leq 1$. Our function is actually $x^2 \sin(1/x)$, so we should multiply through by x^2 to get $-x^2 \leq x^2 \sin(1/x) \leq x^2$. $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$. Then by the Squeeze Theorem, $\lim_{x\to 0} x^2 \sin(1/x) = 0$.*



Example 1.0.4 *Let $f(x)$ be a function such that $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$. What is $\lim_{x\to 4} f(x)$? Let's check the limits on either side! $\lim_{x\to 4} 4x - 9 = \lim_{x\to 4} x^2 - 4x + 7 = 7$, so $\lim_{x\to 4} f(x) = 7$. In the graph below, we have $4x - 9$ and $x^2 - 4x + 7$. $f(x)$ could be any function that stays between those two.*



2 Rationalizing Functions

Example 2.0.5 *Evaluate* $\lim_{x\to 16} \dfrac{\sqrt{x}-4}{x-16}$. *We want to get rid of the square root. Remember that* $(a+b)(a-b) = a^2 - b^2$. *So let's multiply our function by a fancy 1, namely* $\frac{\sqrt{x}+4}{\sqrt{x}+4}$. *This is called the* ***conjugate*** *of* $\sqrt{x}-4$ *because it gets rid of the square root. When we do this, we have* $\lim_{x\to 16} \frac{x-16}{(x-16)(\sqrt{x}+4)}$. *Since we're taking the limit, we can cancel the* $x-16$ *to get* $\lim_{x\to 16} \frac{1}{\sqrt{x}+4} = \frac{1}{8}$.

Example 2.0.6 *Evaluate* $\lim_{x\to 3} \dfrac{x^2-9}{\sqrt{x}-\sqrt{3}}$. *Again, we want to multiply by the conjugate, so we have*

$$\begin{aligned}
\lim_{x\to 3} \frac{(x+3)(x-3)}{\sqrt{x}-\sqrt{3}} \cdot \frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} &= \lim_{x\to 3} \frac{(x+3)(x-3)(\sqrt{x}+\sqrt{3})}{x-3} \\
&= \lim_{x\to 3} (x+3)(\sqrt{x}+\sqrt{3}) \\
&= 6(\sqrt{3}+\sqrt{3}) \\
&= 12\sqrt{3}.
\end{aligned}$$

Practice Problems

See the practice problems we did in class, also linked on the website.